Forces and Motion

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Videos are were provided demonstrating the motion of various systems. Three videos were given in particular, showcasing three systems.

- A cart was pulled by a stretched Hookean spring.
- A fan jet was attached to a cart propelling it forwards.
- A cart in motion splits into two halves.

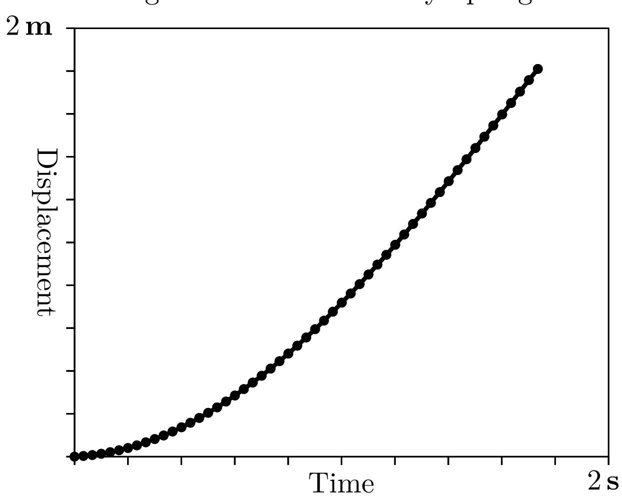
The carts had low friction wheels and moved on a straight and level surface on a track that was parallel to the camera's focal plane. Two points on the track are of a known distance apart, enabling the extraction of distance measurements from the video.

The videos will be tracked using software to extract numerical data about the cart locations over time. Known Newtonian models will be applied to determine meaningful properties about each system.

1 Cart Pulled By Spring

A nearly Hookean spring pulled a cart across the track. Here is the graph of its displacement over time:

Figure 1: Cart Pulled By Spring



1.1 Quadratic Regression

Looking at the graph alone, it can be tempting to use a quadratic regression to model the motion. Such regression would have three parameters:

Parameter	Symbol	Unit
Acceleration	a	Acceleration
Initial Velocity	v_0	Speed
Initial Displacement	Δx_0	Length

The equation using these parameters would be:

$$\Delta \hat{x}(t) = \frac{1}{2}at^2 + v_0 t + \Delta x_0$$

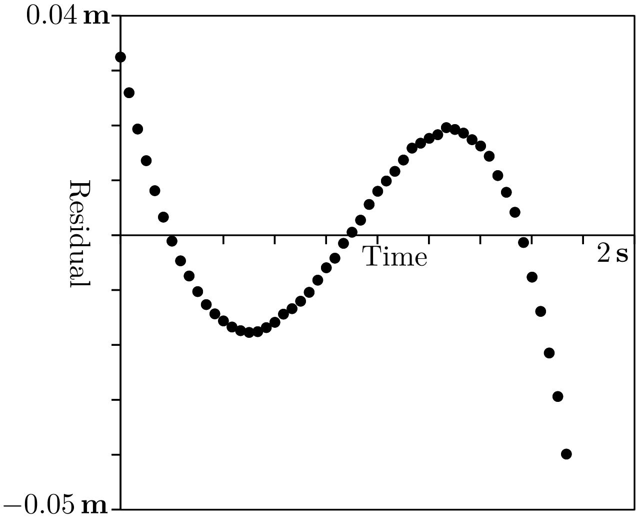
A numerical solver was used to find the optimal parameters to best match the motion curve.

a	$0.932952\,\mathbf{m/s^2}$
v_0	$0.275475\,\mathbf{m/s}$
Δx_0	$-0.032451\,\mathbf{m}$

The residual standard error using this model was $0.016040\,\mathbf{m}$.

Here is the residual plot:

Figure 2: Cart-Spring Quadratic Residuals



An obvious pattern emerges, indicating that a quadratic regression is not a good fit.

1.2 Sinusoidal Regression

Finding a better fit involves considering the forces acting on the cart. The main force acting on the cart was the spring force. The spring began fully extended, meaning the acceleration began at its maximum magnitude. As the cart moved, the spring became less stretched and the acceleration magnitude decreased.

Because the acceleration is negatively proportional to displacement, the displacement over time can be modeled by a sinusoidal function with four parameters:

Parameter	Symbol	Unit
Amplitude	A	Length
Angular Frequency	ω	Frequency
Phase Shift	ϕ	None
Displacement Shift	x_0	Length

These parameters apply to the following equation of estimated displacement over time:

$$\Delta\hat{x}(t) = A\sin(\omega t - \phi) + x_0$$

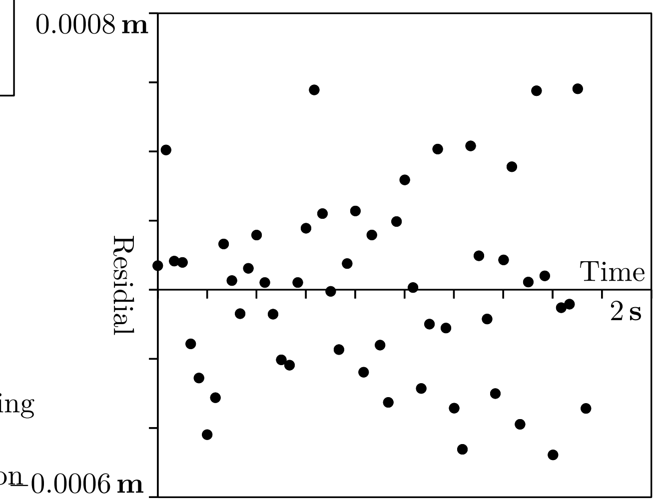
The numerical solver was used to optimize the four parameters.

A	1.78904 **m**
ω	0.88987 **Hz**
ϕ	3.10210
x_0	1.78758 **m**

The residual standard error using this model was 0.000276 **m**, much lower than for the quadratic model.

Here is the residual plot:

Figure 3: Cart-Spring Sinusoidal Residuals



While the chaotic nature of the graph indicates a sinusoidal function is a decent model, the residual values seem to slightly grow in magnitude over time, besides a few initial outliers. This is possibly due to air resistance acting more prominently at higher speeds as well as friction further degrading at the validity of the model.

It is known that for spring-attached objects following perfect sinusoidal motion, the angular frequency of motion is equal to $\sqrt{\frac{k}{M}}$ where k is the spring constant and M is the mass of the object. It was given that the spring constant was in fact 0.90 $\mathbf{N/m}$, and the mass was in fact 1.51 **kg**, resulting in an angular frequency of about 0.77(2028) **Hz**. The angular frequency obtained from video analysis was

0.88987 **Hz** , leaving the percent error around 15.3%. This is not great agreement, but it makes sense considering the inaccuracies of the video and the assumptions made in the analysis process.

As a side comment, the residual plot for the quadratic regression in Figure 2 seems to resemble a cubic function. I wonder if a cubic regression may result in the residual plot looking like a quartic graph, and if any polynomial regression would result in residuals looking similar to a polynomial of one degree higher. My train of thought comes from the fact that any sinusoidal function can be broken into a Taylor Series.

1.3 Friction and Air Resistance

The sinusoidal model neglects air resistance and friction entirely. It is possible to measure the level of air resistance and friction by considering their respective forces.

$$F = -k\left(x - x_0\right) - \mu M g - bv$$

$$a = \frac{-k}{M}\left(x - x_0\right) - \mu g - \frac{b}{M}v$$

v and a can be computed numerically by taking finite differences in displacement Δx. Meanwhile, k/M, x_0, μg, and b/M are constants that are to be found.

The numerical solver yielded these results:

k/M	$0.84031\,\mathbf{Hz}^2$
x_0	$1.75873\,\mathbf{m}$
μg	$0.052133\,\mathbf{m/s^2}$
b/M	$1.3 \cdot 10^{-16}\,\mathbf{Hz}$

b/M is extremely small, showing that air resistance had almost no effect on the motion. The friction term is low, but significant. By assuming $g = 9.81\,\mathbf{m/s^2}$, the coefficient of friction μ can be determined to be about 0.005314, which is expected for a low friction rolling cart.

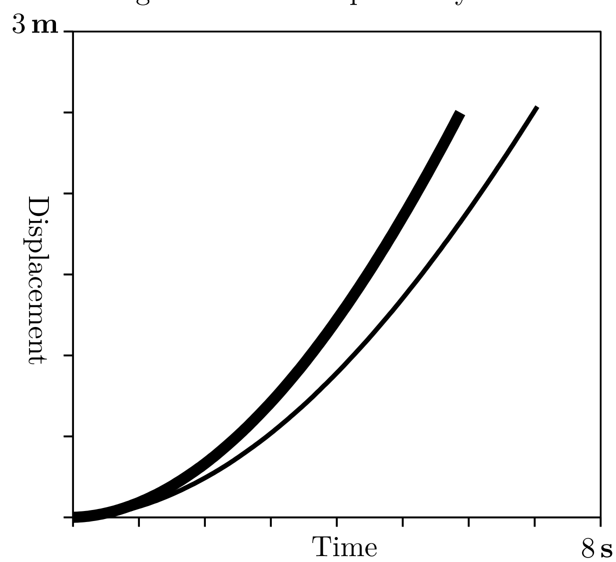
Returning to sinusoidal motion, the angular frequency is known to be $\sqrt{k/M}$, about 0.916668 **Hz** according to these results. This is even more far off from the true known value of 0.77(2028) **Hz** . This extra discrepancy is possibly the result of the fact that finite differences are more sensitive to small errors.

By contrast the value of x_0 agrees with the value found in the sinusoidal regression. This could be related to the fact that x_0 is a value that is subtracted from x, while k/M is a scaling value, which makes it intuitively seem more sensitive to error.

2 Cart Propelled By Fan

A fan jet was attached to a cart, propelling it forward. Two trials were performed, each at different fan speeds. Below is the graph of displacement over time. The thin line is the low speed trial, and the bold line is the high speed trial.

Figure 4: Cart Propelled By Fan



2.1 Quadratic Regression

The graphs appear to look somewhat quadratic in nature. Quadratic motion would suggest that the fan results in a constant force on the cart. A quadratic regression was performed to optimize the equation $\Delta x = \frac{1}{2}at^2 + v_0 t + \Delta x_0$.

Table 1: Quadratic Regression: Slower Fan

a	$0.0457670\,\mathrm{m/s^2}$
v_0	$0.0439673\,\mathrm{m/s}$
Δx_0	$-0.0196277\,\mathbf{m}$

The residual standard error for the slow speed fan was $0.0008412\,\mathbf{m}$.

Table 2: Quadratic Regression: Faster Fan

a	$0.0658992\,\mathrm{m/s^2}$
v_0	$0.0436374\,\mathrm{m/s}$
Δx_0	$-0.0138828\,\mathbf{m}$

The residual standard error for the slow speed fan was $0.0003753\,\mathbf{m}$.

Looking at the parameters alone, it can be reasoned that the cart with the faster fan accelerates at a higher rate.

Here are the residual plots for each regression:

Figure 5: Slow Cart-Fan Quadratic Residuals

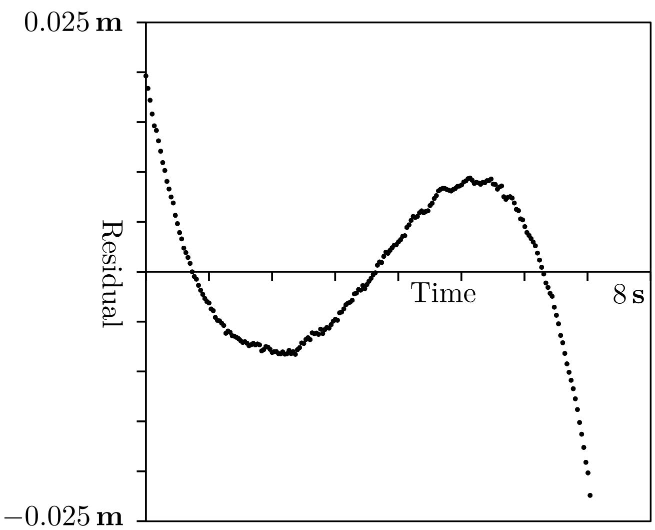
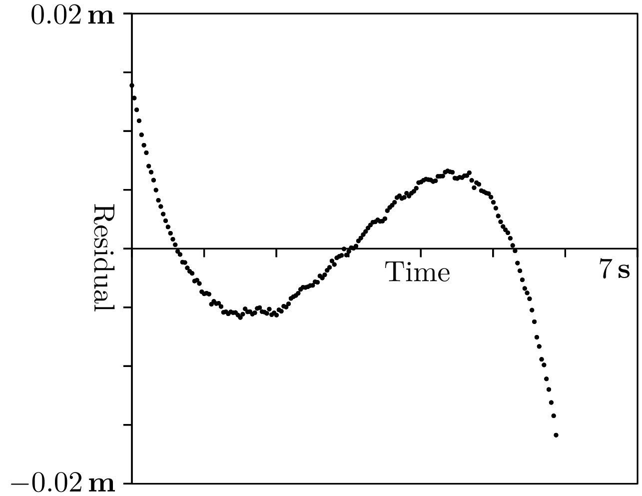


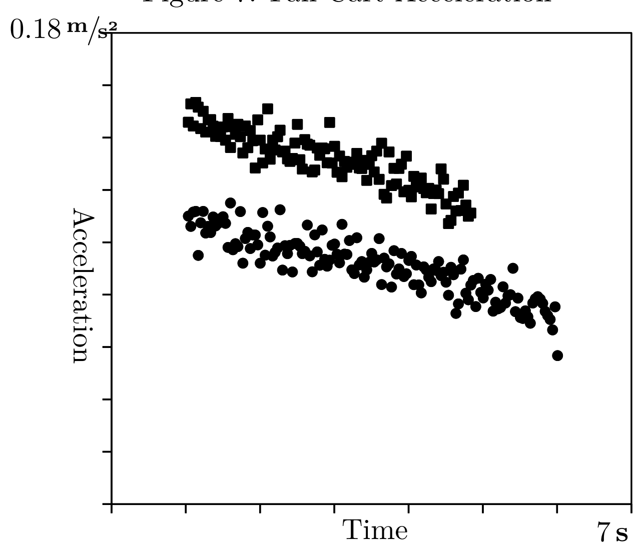
Figure 6: Fast Cart-Fan Quadratic Residuals



A clear pattern emerges from the residual plots, suggesting that a quadratic regression is not a good fit for the motion of the fan-propelled cart.

This suggests that the cart's acceleration changes over time is not constant. A rough graph of acceleration over time can be made from finite differences in position over time. Below is such a graph. A moving average over 2 seconds had to be applied to make the results intelligible.

Figure 7: Fan-Cart Acceleration



The circles are for the slow fan while the squares are for the fast fan.

The acceleration appears to decrease over time. Additionally, the acceleration for the fast fan cart is confirmed to be higher than for the slow fan cart.

2.2 Force Analysis

The fan moves air mass past it at some rate $\frac{dm}{dt}$. By Newton's Second Law, the air experiences some force $\frac{dp}{dt}$, which by the product rule, equals $m\frac{dv}{dt} + v\frac{dm}{dt}$, where m represents the total mass pulled through and v is the air's velocity relative to the ground. Letting u be the speed of the air relative to the *cart*, given the cart is moving at some velocity V, the force f acting on the air from the fan is:

$$f = ma + (u - V)\dot{m}$$

By Newton's Third Law, the force exerted by the fan onto the air is equal to the opposite of the force the air exerts on the fan/cart. Thus, with capital letters representing cart attributes and lowercase letters representing air attributes,

$$MA = ma + (u - V)\dot{m}$$

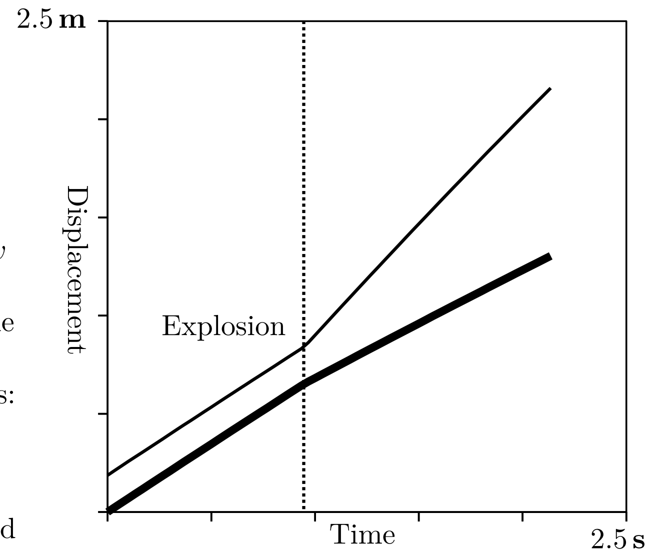
$$A = \frac{1}{M}\left(ma + \dot{m}u - \dot{m}V\right)$$

This result suggests that if air acceleration and cart velocity were both negligible compared to cart velocity, and the fan speed was constant, the cart would move with roughly constant acceleration. As demonstrated by the invalidity of the quadratic regressions above, this is not the case. Thus, it can be concluded that either the air accelerates significantly, the cart moves significantly compared to airspeed, or a combination of both factors.

3 Exploding Moving Cart

A set of two carts were firmly attached and set into motion. During motion, the carts each exerted an impulse on each other, causing the cart pair to separate into two. The carts moved rightward, the left cart had mass 1.995 **kg**, and the right cart had mass 0.498 **kg**. Here is the graph of their motion:

Figure 8: Exploding Cart Motion



3.1 Quadratic Regression

The only external force acting on the carts is friction. Otherwise, the force between the carts is the impulse of explosion, which is assumed to happen so quickly that it might as well be considered as an instantaneous velocity change for each cart.

The motion of each cart is a piecewise quadratic function. Let the heavier cart inherit capital letters and the lighter cart inherit lowercase letters. Let X_0, x_0 represent initial position, v_0 represent initial velocity pre explosion, V_1, v_1 represent initial velocity post explosion, μ represent the coefficient of friction, and t_e represent the time of explosion.

$$\Delta X = \begin{cases} X_0 + v_0 t + \frac{1}{2}\mu g t^2 & t \le t_e \\ X_0 + v_0 t_e + V_1 t + \frac{1}{2}\mu g \left(t^2 + t_e^2\right) & t \ge t_e \end{cases}$$

$$\Delta x = \begin{cases} x_0 + v_0 t + \frac{1}{2}\mu g t^2 & t \le t_e \\ x_0 + v_0 t_e + v_1 t + \frac{1}{2}\mu g \left(t^2 + t_e^2\right) & t \ge t_e \end{cases}$$

A numerical solver was used to solve for all of the unknown parameters so the curves best match the data. Here are the results:

t_e	$0.9365977\,\mathbf{s}$
X_0	$-0.002280\,\mathbf{m}$
x_0	$0.1856744\,\mathbf{m}$
v_0	$0.7098115\,\mathbf{m/s}$
V_1	$0.5753316\,\mathbf{m/s}$
v_1	$1.1366653\,\mathbf{m/s}$
μ	0.0044034

The standard residual error for the heavy and light cart displacement models are $0.99 \cdot 10^{-5}\,\mathbf{m}$ and $2.24 \cdot 10^{-5}\,\mathbf{m}$ respectively.

Here are the residual plots for each cart:

Figure 9: Heavy Cart Residuals

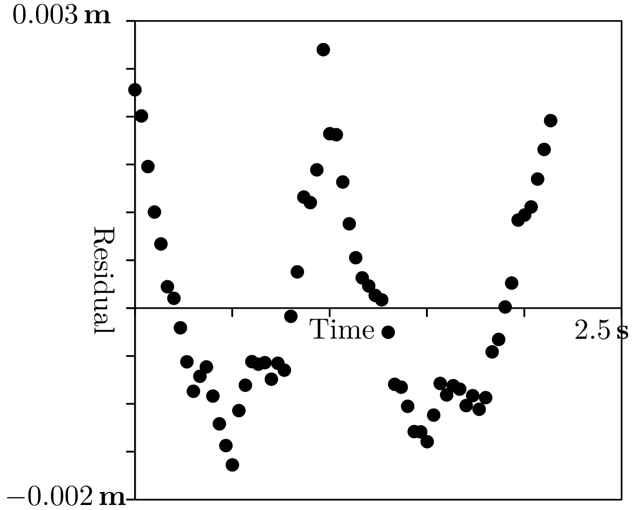
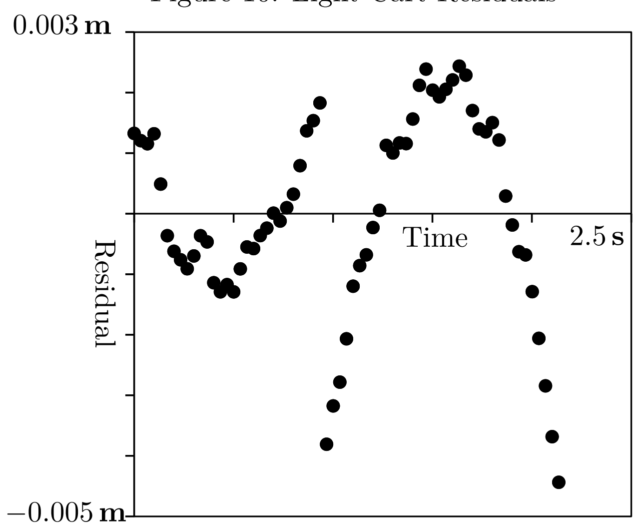


Figure 10: Light Cart Residuals



The clear quadratic nature of the residuals suggests that the coefficient of friction may be different for each cart, or some other force like air resistance was also present.

3.2 Conservation of Momentum

The momentum change of the system over time should be equal to the sum of external forces on it, which in this case is friction only. The internal force of the cart explosion should have no impact on the momentum of the system. Thus, the limit of the momentum as time approaches the explosion instant should be the same on the left and right hand sides.

$$p = MV + mv = M\frac{d\Delta X}{dt} + m\frac{d\Delta x}{dt}$$

Going by the model found above, the derivatives shown here can be analytically found.

$$V = \begin{cases} v_0 + \mu g t & t \le t_e \\ V_1 + \mu g t & t \ge t_e \end{cases}$$

$$v = \begin{cases} v_0 + \mu g t & t \le t_e \\ v_1 + \mu g t & t \ge t_e \end{cases}$$

The limit of the piecewise function at the point between two pieces of it is equivalent to the evaluation of those pieces at specific instances. Thus,

$$\lim_{t \to t_e^-} p = (M + m)(v_0 + \mu g t_e)$$

$$\lim_{t \to t_e^+} p = M(V_1 + \mu g t_e) + m(v_0 + \mu g t_e)$$

The numerical values for both left and right limits are known.

$$\lim_{t \to t_e^-} p = 1.870(423)\,\mathbf{kg \cdot m/s}$$

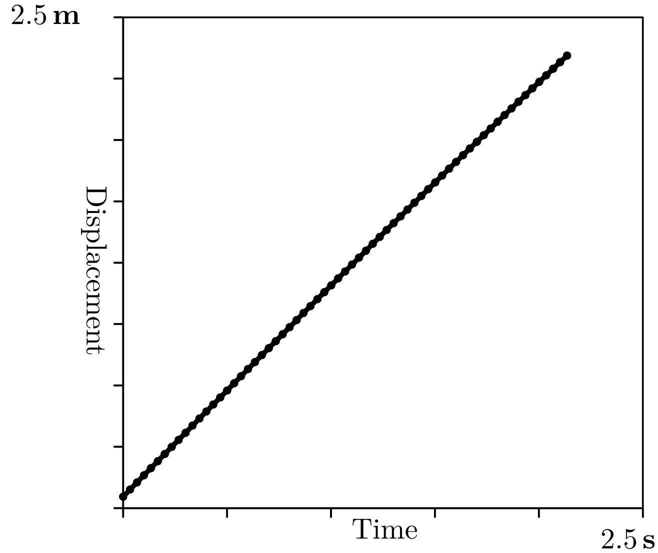
$$\lim_{t \to t_e^+} p = 1.814(709)\,\mathbf{kg \cdot m/s}$$

These two values are very close, suggesting that the momentum change due to the explosion is very small. What little momentum change there is could be a result of the explosion spring being inefficient or air resistance being more apparent during the explosion.

3.3 Center Of Mass

The center of mass of a system acts like a particle experiencing the sub of external forces experienced on each component of the system. Thus, due to friction we expect the center of mass to follow a quadratic motion. Here is the graph of the center of mass over time:

Figure 11: Motion of Center Of Mass



The graph looks nearly linear because friction is very low. The continuity of the graph demonstrates that the center of mass moves continuously with constant, nearly negligible force. The explosion of the system barely affected the motion of the center of mass, as the time of explosion cannot even be distinguished in this graph from its features alone.