Rolling Motion

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When a round object rolls down a decline, the amount by which the object's surface slips on the ramp can vary. Here, the degree of slippage will be analyzed and compared to attributes like decline angle, friction, and rotational inertia.

1 Theory

In a Newtonian sense, our rolling object experiences three forces: gravity, normal forces from the ramp, and friction against the ramp. If ϕ is the angle the ramp makes with the horizontal, g is gravitational acceleration, and f is the frictional force, then a rolling object of mass M, moment of inertial I, radius R, acceleration a, and rotational acceleration α is described by the following equations:

$$Ma = Mg\sin\phi - f$$

$$I\alpha = -Rf$$

If the object is slipping, the frictional force is kinetic in nature and its magnitude is a constant multiple of the normal force. Given coefficient of friction μ, in the case of slipping,

$$\boxed{f = \mu Mg\cos\phi}$$

Thus in the case of slipping, a and α are such that:

$$\boxed{a = g\sin\phi - \mu g\cos\phi}$$

$$\boxed{\alpha = -\frac{1}{I}R\mu Mg\cos\phi}$$

If the object is not slipping, the acceleration and angular acceleration are geometrically related by the equation

$$a = -\alpha R$$

In the event of no slipping, the above equation can be substituted into the equations of motion.

$$Ma = Mg\sin\phi - f \implies f = Mg\sin\phi - Ma$$

$$I\frac{-a}{R} = -Rf \implies f = \frac{Ia}{R^2}$$

$$Mg\sin\phi - Ma = \frac{Ia}{R^2}$$

$$a = \frac{Mg\sin\phi}{M + \frac{I}{R^2}} = \boxed{a = \frac{MR^2 g\sin\phi}{MR^2 + I}}$$

$$f = \frac{Ia}{R^2} = \boxed{f = \frac{MIg\sin\phi}{MR^2 + I}}$$

$$\alpha = \frac{-a}{R} = \boxed{\alpha = \frac{-MRg\sin\phi}{MR^2 + I}}$$

Additionally, in the case of no slippage, the frictional force is static in nature and its magnitude is bounded such that $f \leq \mu Mg\cos\phi$. Thus,

$$\frac{MIg\sin\phi}{MR^2 + I} \leq \mu Mg\cos\phi$$

$$\tan\phi \leq \frac{\mu}{I}(MR^2 + I)$$

Since $0 \leq \phi \leq \frac{(2\pi)}{4}$, tan for this range is injective. Thus,

$$\phi \leq \arctan\left(\frac{\mu}{I}(MR^2 + I)\right)$$

$$\boxed{\text{Let } \beta = \arctan\left(\tfrac{\mu}{I}(MR^2 + I)\right).}$$

$$\phi \leq \beta$$

Therefore, the object is not slipping if and only if $\phi \leq \beta$. Thus the object *is* slipping if and only if $\phi > \beta$. β is the critical angle.
Furthermore, a sensible unitless quantity can be derived by comparing a to $R\alpha$. When there is no slip,

$$\boxed{\frac{a}{R\alpha} = -1}$$

When there is slip,

$$\frac{a}{R\alpha} = \frac{-Ig\sin\phi + I\mu g\cos\phi}{R^2\mu Mg\cos\phi} = \boxed{\frac{a}{R\alpha} = \frac{I}{MR^2}\left(1 - \frac{1}{\mu}\tan\phi\right)}$$

2 Experiment

The above expressions can be confirmed experimentally. A thick cylindrical shell was rolled down a decline at several different angles.

The shell had a mass of $\underline{0.452\,\mathbf{kg}}$.

It had an outer radius of $\underline{0.0564\,\mathbf{m}}$.

Based on the above and its inner radius of $0.0482\,\mathbf{m}$, the rotational inertial was calculated to be $\underline{0.00124(395)\,\mathbf{kg}\cdot\mathbf{m^2}}$.

2.1 Acceleration

As the cylinder was set into rolling motion by gravity down the decline, the magnitude of its displacement and the total angle moved was tracked. From this data over time, a quadratic regression was performed. This process was repeated for each angle of decline. Using the regression data, both the linear acceleration and angular acceleration were extracted. Below, the error shown is for a 96.875% confidence interval.

Figure 1: Linear Acceleration

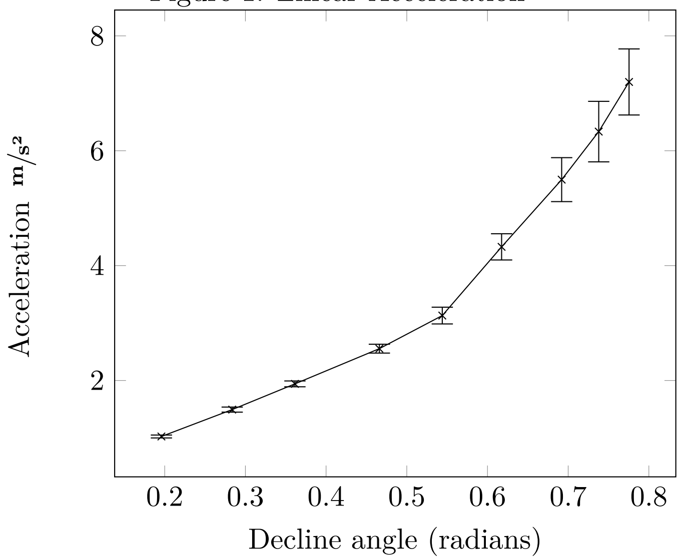
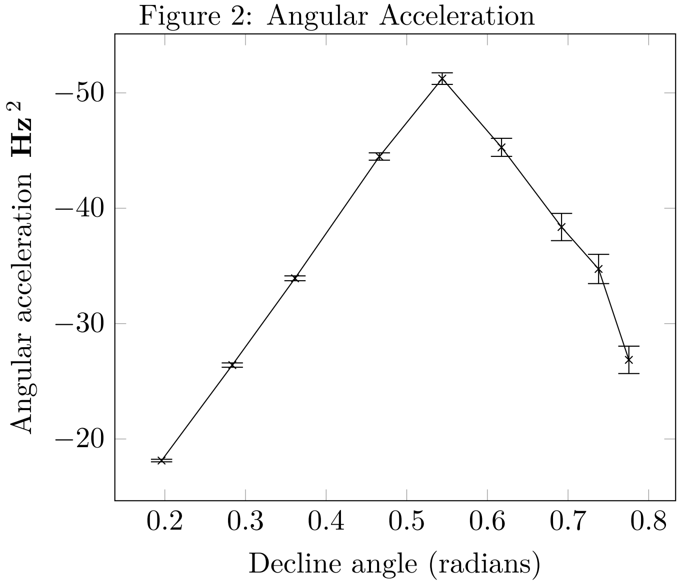


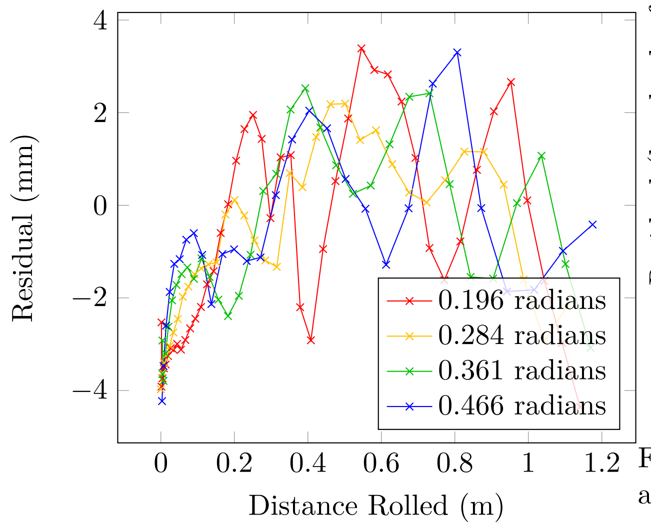
Figure 2: Angular Acceleration



2.2 Regression residuals

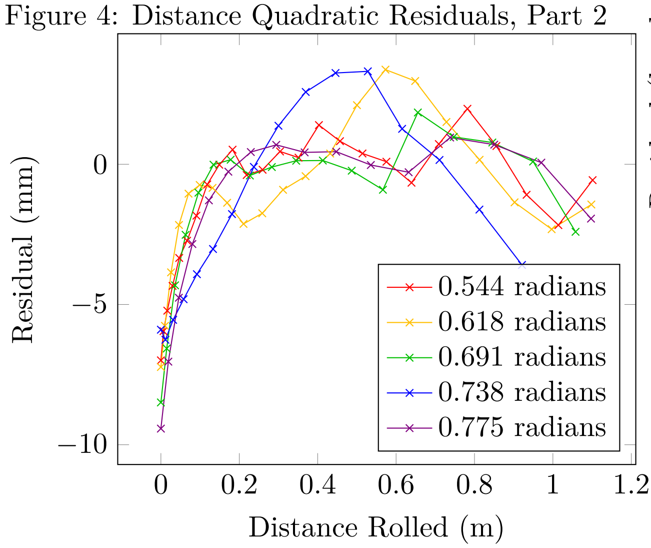
The regression residual plots reveal that acceleration is not exactly constant. There is some oscillation which is reminiscent of the rotation of the cylinder. It can be reasoned that the point of motion tracking for each trial was not exactly centered.

Figure 3: Distance Quadratic Residuals, Part 1



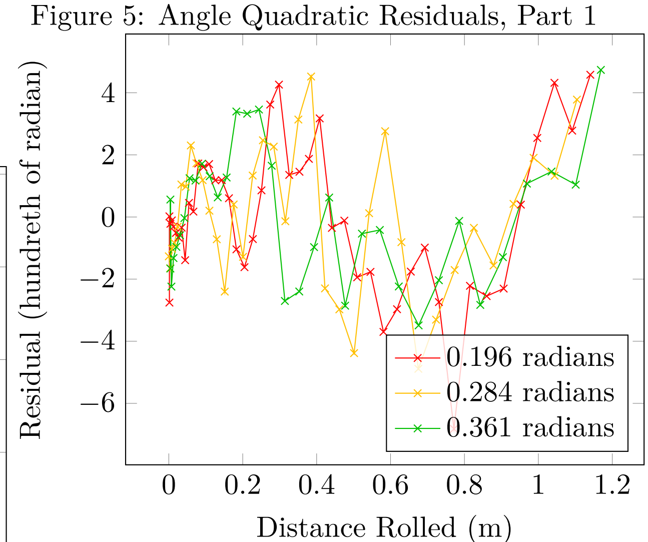
For steeper angles, there is an apparent skew in the residuals. This suggests that a variable force is present, which could be air resistance or something else. Even so, these residual values are only on the order of millimeters, and the motion is well correlated to a quadratic curve.

Figure 4: Distance Quadratic Residuals, Part 2



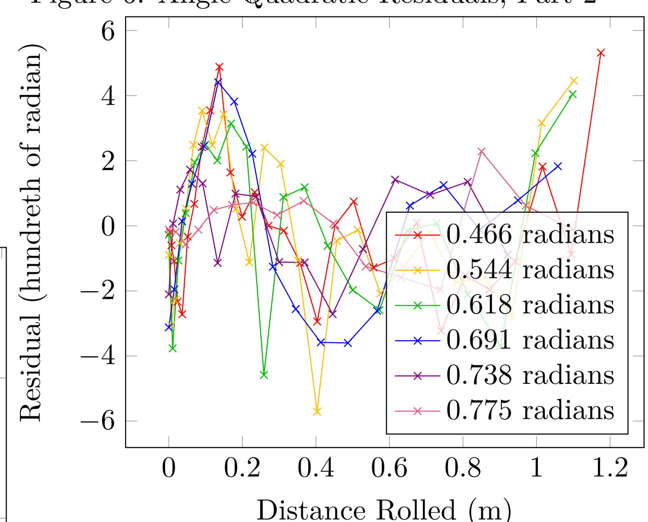
The regression residuals for angle over time show a bit of a skew. A mild oscillation in phase with the rotation is seen, suggesting the cylinder may be slightly unbalanced.

Figure 5: Angle Quadratic Residuals, Part 1



For steeper angles, the cylinder did not rotate as much, hence a longer period for the slight oscillation.

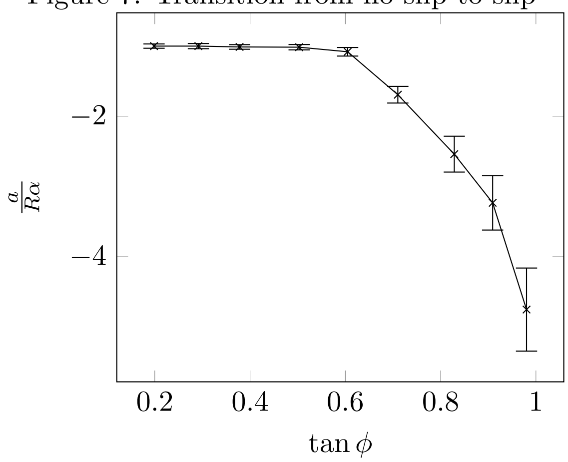
Figure 6: Angle Quadratic Residuals, Part 2



3 Experiment vs. Model

The degree and possible sources of error have been identified. In all, the quadratic regressions describe the motion of the cylinder quite well. The r^2 values for each quadratic regression were no lower than 0.9985, with only a few outliers below 0.9995. This suggests that the linear and angular accelerations of the cylinder are constant. As shown by the model derived before, the unitless quantity $\frac{a}{R\alpha}$ is constant below the critical angle and linear with $\tan\phi$ above. Thus, the transition between no slipping and slipping can be illustrated by identifying a change in slope of the graph of $\frac{a}{R\alpha}$ over $\tan\phi$.

Figure 7: Transition from no slip to slip



The graph agrees with the model; the magnitude of $\frac{a}{R\alpha}$ is indeed 1 before the critical angle.

The point at $\phi = 0.544$ is such that the error bar misses the value -1. Thus it can be concluded that the critical angle is between 0.466 and 0.544.

The formula for the critical angle and the formulas for $\frac{a}{R\alpha}$ before and slip shows that precisely at the critical angle, both formulas agree at -1. Therefore a linear regression can be performed on the slipping half of the data, and its intersection with the value -1 should lie at the value of the tangent of the critical angle.

This regression was performed to determine that the critical angle is 0.558 ± 0.126 (with 96.875% confidence). However, this range completely subsumes the range between 0.466 and 0.544, suggesting that there is simply not enough data to precisely calculate the critical angle.

If the critical angle was known for certain, the coefficient of friction could be calculated using the critical angle's formula.

$$\mu = \frac{I \tan\beta}{MR^2 + I}$$

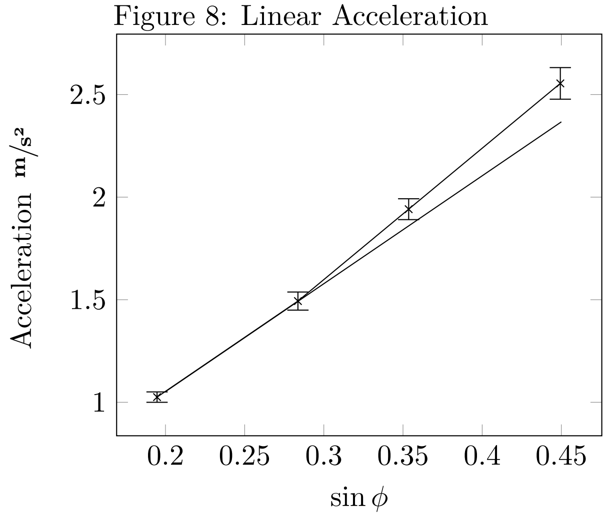
If the critical angle really was 0.558, the coefficient of friction would be about 0.289.

Below the critical angle, the acceleration should be proportional to $\sin\phi$. Specifically,

$$a = \frac{MR^2 g}{MR^2 + I}\sin\phi$$

So in our case, the acceleration should be proportional to $\sin\phi$ by a factor of $5.260\,\mathrm{m/s^2}$. Below, the true acceleration is compared to $5.260\,\mathrm{m/s^2}\,\sin\phi$.

Figure 8: Linear Acceleration



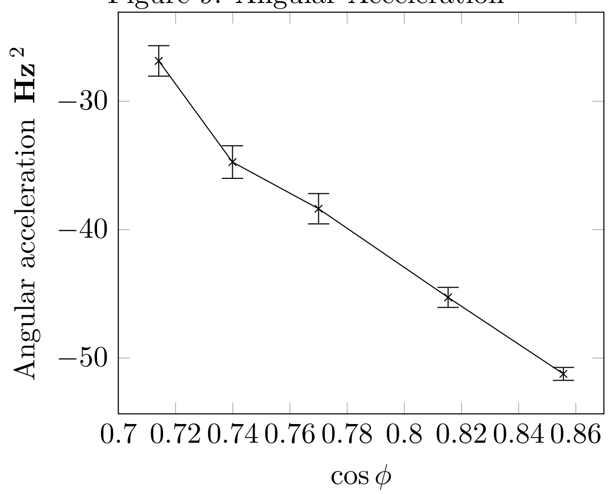
It is visible that the results don't entirely agree. This could suggest that the video data was inaccurate or incorrectly measured, or possibly that the moment of inertia was actually higher than calculated.

Beyond the critical angle, angular acceleration should be proportional to $\cos\phi$.

$$\alpha = -\frac{1}{I}R\mu Mg\cos\phi$$

In our case the angular acceleration should be proportional to $\cos\phi$ by a factor of $-201.041\mu\,\mathbf{Hz}^2$.

Figure 9: Angular Acceleration



The slope found by linear regression is $-163.361\,\mathbf{Hz}^2 \pm 105.221\,\mathbf{Hz}^2$, leaving the value of μ to be in the range of 0.813 ± 0.523. The error is so large that this result has little significance; it is very plausible for the true coefficient to lie anywhere within this large range.

4 Conclusion

The model successfully described the cylinder rolling down the ramp. While the exact coefficient of friction and critical could not be calculated, ranges were found and it was demonstrated that with more precise and numerous data, these values could be determined precisely.